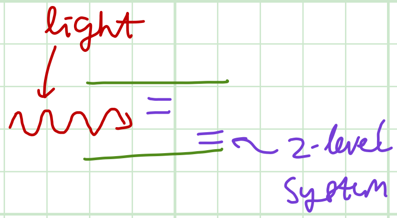
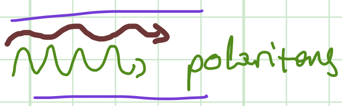
Plan: – Semiclassical model of polaritons
– "quantum" polaritons

The physical point of the picture:

Consider shining light on a crystal

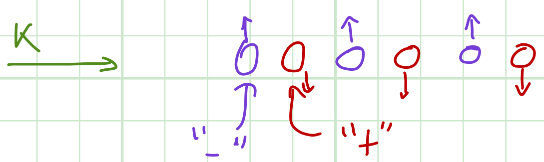


Old picture: light propagates, sometimes being absorbed or reflected by the oscillators / 2-level systems inside the crystal



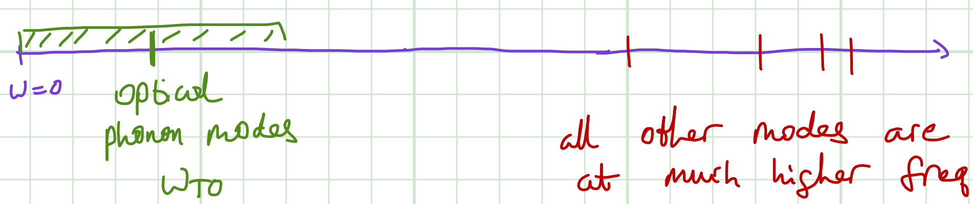
New picture: light modes + polarization modes couple to produce a new set of modes.
$\Rightarrow$ the polariton modes that combine the characteristics of the components.

optical phonons $\Rightarrow$ must have solid with at least 2 distinct atoms per unit cell



let's couple to the photons $\Rightarrow$ semi-classical picture

Susceptibility of a medium
range of ω we want.



$\Rightarrow \omega_\infty$

$$\varepsilon(\omega) = \varepsilon_0\left(1+\chi_{TO}(\omega) + \tilde{\chi}(\omega)\right)$$

the transverse phonons

higher freq. modes

where $\chi_{TO}(\omega) = \frac{q^2 N}{\varepsilon_0 m_r V} \frac{1}{\omega_{TO}^2 - \omega^2}$

q = effective charge displaced

m_r = effective mass

ω_{TO} = phonon freq [this used to be ω_0]

and $\tilde{\chi} \sim \frac{1}{\omega_\infty^2 - \omega^2} \sim \frac{1}{\omega_\infty^2} = \text{const} \equiv (\varepsilon(\infty) - 1)/\varepsilon_0$

Let us reparametrize our expression for χ_T

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \tilde{\chi} + \frac{C_1}{1 - \omega^2/\omega_{TO}^2}\right) = \varepsilon(\infty) + \frac{C_1 \varepsilon_0}{1 - \omega^2/\omega_{TO}^2}$$

For $\omega \to 0$, $\varepsilon(\omega) \to \varepsilon(0)$. Let us use $\varepsilon(0)$ to parametrize C_1.

$$\varepsilon(\omega \to 0) = \varepsilon(\infty) + \frac{C_1 \varepsilon_0}{1 - 0/\omega_T^2} \Rightarrow C_1 \varepsilon_0 = \varepsilon(0) - \varepsilon(\infty)$$

$$\boxed{\Rightarrow \varepsilon(\omega) = \varepsilon(\infty) + \frac{\varepsilon(0) - \varepsilon(\infty)}{1 - \omega^2/\omega_{TO}^2}}$$

this expression is valid for $\omega \ll \omega_\infty$

Transverse vs. longitudinal photons

Remember that in medium, we we have $\nabla \cdot D = 0$, where $D \equiv \varepsilon(\omega) E$.

There are two ways to satisfy this Relation:

(1) Demand that $\nabla \cdot E = 0$

plug in a wave-like solution $\vec{E} = \vec{E}_0 e^{i(kx - \omega t)}$

$$\nabla \cdot E = ik \cdot E_0 e^{i(kx - \omega t)} = 0$$

$\Rightarrow E \perp K$ i.e. to direction of propagation of light

This is called a "transverce" photon

This type of photon must satisfy the Maxwell's wave equation:

$$\frac{\partial D}{\partial t} = \nabla \times H \quad \frac{\partial B}{\partial t} = -\nabla \times E \quad \nabla \cdot E = 0 \quad \nabla \cdot B = 0 \quad \Rightarrow \varepsilon \mu_0 \frac{\partial^2 E}{\partial t^2} = \nabla^2 E$$

(2) Demand $\varepsilon(\omega) = 0$

$\Rightarrow \nabla \cdot D = 0$, but $\nabla \cdot E \neq 0$

Hence, $\vec{E}$ has a component along the direction of propagation

This is called a "longitudinal" photon

note: longitudinal photons do not satisfy Maxwell's wave eq:

since $\nabla\times\nabla\times E = \nabla(\nabla\cdot E) - \nabla^2 E = \nabla^2 E$

Let us now obtain the dispersion of these potons – phonons:

Transverse case:

Let us plug this expression for the dielectric fnc. into the Maxwell's wave equation $\omega^2\,\varepsilon(\omega) = (c\,q)^2$

$$\omega^2\left(\varepsilon(\infty) + \frac{\varepsilon(0)-\varepsilon(\infty)}{1-\omega^2/\omega_{TO}^2}\right) = (cq)^2$$

Now solve for ω as a function of q

this is just a quadratic eqn for ω^2:

$$\omega^2 = \frac{1}{2}\left[\frac{\varepsilon(0)\,\omega_{TO}^2}{\varepsilon(\infty)} + \tilde{\omega}_q^2\right] \pm \frac{1}{2}\left[\left(\tilde{\omega}_q^2 + \frac{\varepsilon(0)\,\omega_{TO}^2}{\varepsilon(\infty)}\right)^2 - 4\,\tilde{\omega}_q^2\,\omega_{TO}^2\right]^{1/2}$$

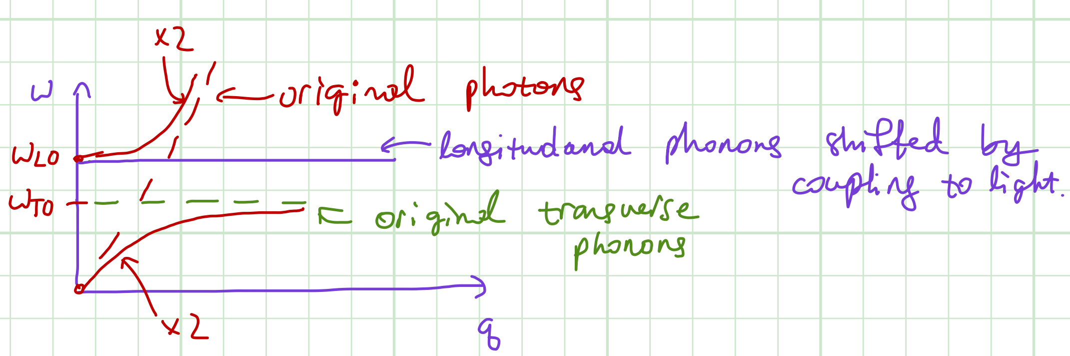
where $\tilde{\omega}_q^2 = \dfrac{(cq)^2}{\varepsilon(\infty)}$

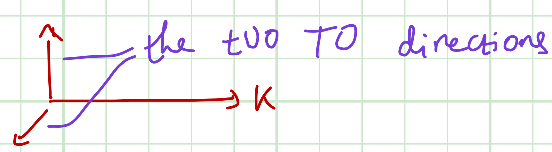
What does this solution mean? Let's plot

$$q\to 0 \Rightarrow \omega^2 = \frac{\varepsilon(0)\,\omega_{TO}^2}{2\,\varepsilon(\infty)} \pm \frac{1}{2}\left(\frac{\varepsilon(0)\,\omega_{TO}^2}{\varepsilon(\infty)}\right)$$

$$= \left\{0,\ \sqrt{\frac{\varepsilon(0)\,\omega_{TO}^2}{\varepsilon(\infty)}}\right\}$$

This is a new freq, we will see that it matches ω_{LO}, which makes sense as there is no distinction for $k=0$.





longitudinal case:

We demand that $\varepsilon(\omega_{LO}) = 0$.

Instead of solving for ω_{LO}, I just plug-in the value for ω_{TO} at $k=0$

$$\varepsilon(\omega_{LO}) = \varepsilon(\infty) + \frac{\varepsilon(0)-\varepsilon(\infty)}{1-\frac{\varepsilon(0)}{\varepsilon(\infty)}\frac{\omega_{TO}^2}{\omega_{TO}^2}} = \varepsilon(\infty) + \frac{(\varepsilon(0)-\varepsilon(\infty))\,\varepsilon(\infty)}{\varepsilon(\infty)-\varepsilon(0)} = 0$$

Did we get the correct # of modes?

$\Rightarrow$ Counting of modes

"free" theory		"coupled" theory
phonons: 2 TO 1 LO	$\parallel$	1 shifted LO
photons: 2 (transverse)		2 upper branch polaritons
		2 lower branch polaritons

($\checkmark$)

Propagation of light in medium

$\Rightarrow$ no modes between $\omega_{TO} < \omega < \omega_{LO}$

$\rightarrow$ what does this mean?

(1) if I shine a laser of the forbidden freq. onto the sample? can the laser light propagate into the sample?

(2) what is k inside the sample?

(3) what about T + R

The quantum theory of TO polaritons.

Let us focus on just one of the TO polarizations, just like we did in the semiclassical theory.

$$H = \underset{\uparrow a_K}{H_{photon}} + \underset{\uparrow c_K}{H_{phonon}} + H_{int}$$

$$H_{photon} = \sum_K \hbar c|K|\, a_K^+ a_K = \sum_K \hbar \omega_K a_K^+ a_K$$

$$H_{phonon} = \sum_K \hbar \omega_0 c_K^+ c_K$$

To work out $H_{int} = -\hat{P}\cdot\hat{E}$, we first need to work out $\hat{P}$ and $\hat{E}$ operators

Remembering how 2^{nd} quantization works for photons, we write down:

$$\hat{E} = -i \sum_K \vec{n}_K \sqrt{\frac{\hbar\omega_K}{2\varepsilon_\infty V}} \left[a_K e^{ik\cdot r} - a_K^+ e^{-ik\cdot r}\right]$$

$\vec{n}_K$: polarization direction

ε_∞: dielectric constant ignoring contribution from phonons, which we will add explicitly

$$\hat{P}(r) = \frac{N}{V} q\, \hat{X}(r) = \frac{N}{V} q \sqrt{\frac{\hbar}{2m\omega_0}} \left(c^+(r) + c(r)\right) = \frac{N}{V} q \sqrt{\frac{\hbar}{2mN\omega_0}} \sum_K \left(c_K^+ e^{-ik\cdot r} + c_K e^{ik\cdot r}\right)$$

$\hat{X}(r)$: displacement of the phonon at position r.

$$H_{int} = i\hbar\Omega \sqrt{\frac{\omega_K}{\omega_0}} \frac{1}{V} \sum_{KK'} \int d^3r \left(a_K e^{ik\cdot r} - a_K^+ e^{-ik\cdot r}\right)\left(c_{K'} e^{ik'\cdot r} + c_{K'}^+ e^{-ik'\cdot r}\right)$$

$\Omega = \sqrt{\frac{q^2 N}{4\varepsilon_\infty m V}}$

↳ use the fact that $\frac{1}{V}\int d^3r\, e^{iq\cdot r} = \delta_q$ ← "Kronecker delta"

Hence, we find that the total Hamiltonian is

$$\mathcal{H} = \mathcal{H}_{photon} + \mathcal{H}_{phonon} + \mathcal{H}_{int}$$

$$= \sum_K \left[\hbar\omega_K a_K^+ a_K + \hbar\omega_0 c_K^+ c_K + i\hbar\Omega\sqrt{\frac{\omega_K}{\omega_0}}\left(a_K c_K + a_K c_K^+ - a_K^+ c_K - a_K^+ c_K^+\right)\right]$$

This is a quadratic Hamiltonian — all terms are proportional to operators squared, which is nice. However, it has terms like $a_K c_K$ which mix different types of operators. We would rather get rid of these terms to solve the Hamiltonian. We do this by a linear transformation on the operators

$$\xi_K^i = \alpha_K^i a_K + \beta_K^i c_K + \gamma_K^i a_{-K}^+ + \delta_K^i c_{-K}^+$$

[there are two ξ_K^1, ξ_K^2 that correspond to the two original operators a_K and c_K] [This is a generalized Bogoliubov transform.]

$\Rightarrow$ short-cut $\Rightarrow$ write H as a matrix

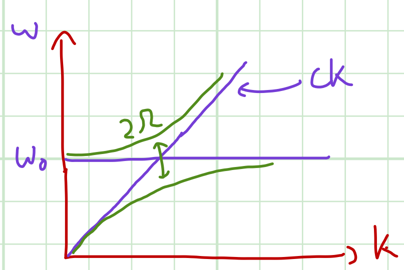
$$\begin{array}{c|cccc|}
 & a_K & c_K & a_{-K}^+ & c_{-K}^+ \\
\hline
c_K^+ & \omega_K & -i\Omega & 0 & -i\Omega \\
c_K^+ & i\Omega & \omega_0 & -i\Omega & 0 \\
a_{-K} & 0 & i\Omega & \omega_K & i\Omega \\
c_{-K} & i\Omega & 0 & -i\Omega & \omega_0
\end{array}
\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \omega \\ \omega \\ -\omega \\ -\omega \end{pmatrix}$$

$\Rightarrow$ these signs are needed to ensure commutation relations between the new operators are satisfied.

$\Rightarrow$ I think D.S. has an error in equation for ω, p. 351. $-4\Omega^2\omega_K^2 \Rightarrow -4\Omega^2\omega_K\omega_0$

$$\Rightarrow \mathrm{Det}\left[H - \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & -\omega & \\ & & & -\omega \end{pmatrix}\right] = \omega^4 - \omega^2(\omega_0^2 + \omega_K^2) + \omega_0^2\omega_K^2 - 4\Omega^2\omega_0\omega_K = 0$$

Plot the roots:



$\Omega \to 0 \Rightarrow$

$$\omega^4 - \omega^2(\omega_0^2 + \omega_K^2) + \omega_0^2\omega_K^2 = 0$$

$$\omega^2 = \frac{\omega_0^2 + \omega_K^2 \pm \sqrt{(\omega_0^2 + \omega_K^2)^2 - 4\omega_0^2\omega_K^2}}{2}$$

$$= \frac{\omega_0^2 + \omega_K^2 \pm |\omega_0^2 - \omega_K^2|}{2}$$

$$= \{\omega_0^2, \omega_K^2\}$$

$\omega_K = \omega_0 \Rightarrow$

$$\omega^4 - \omega^2 2\omega_0^2 + \omega_0^4 - 4\Omega^2\omega_0^2 = 0$$

$$[\omega^2 + \omega_0(\omega_0 + 2\Omega)][\omega^2 + \omega_0(\omega_0 - 2\Omega)] = 0$$

$$\omega = \sqrt{\omega_0(\omega_0 \pm 2\Omega)}$$

$$\omega \sim \omega_0 \pm \Omega \quad \text{for} \quad \omega_0 \gg \Omega$$

Bogoliubov transforms, quick example:

$$\begin{array}{c|cc} & a_k & a^+_{-k} \\ \hline a^+_k & w_k & \Delta \\ a_{-k} & \Delta^* & w_k \end{array}$$

$$a_k = u_k \xi_k + v_k \xi^+_{-k}$$

$$a^+_k = u_k \xi^+_k + v_k \xi_{-k}$$

$$\Rightarrow \quad \begin{pmatrix} a_k \\ a^+_{-k} \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \xi_k \\ \xi^+_{-k} \end{pmatrix}$$

$$\begin{pmatrix} a^+_k & a_{-k} \end{pmatrix} = \begin{pmatrix} \xi^+_k & \xi_{-k} \end{pmatrix} \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix}$$

$$\begin{pmatrix} a^+_k & a_{-k} \end{pmatrix} \begin{pmatrix} w_k & \Delta \\ \Delta^* & w_k \end{pmatrix} \begin{pmatrix} a_k \\ a^+_{-k} \end{pmatrix} = \begin{pmatrix} \xi^+_k & \xi_{-k} \end{pmatrix} \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} w_k & \Delta \\ \Delta^* & w_k \end{pmatrix} \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \xi_k \\ \xi^+_{-k} \end{pmatrix} = \begin{pmatrix} \xi^+_k & \xi_{-k} \end{pmatrix} \begin{pmatrix} \varepsilon_0 & \\ & \varepsilon_0 \end{pmatrix} \begin{pmatrix} \xi_k \\ \xi^+_{-k} \end{pmatrix}$$

Hence: $$\begin{pmatrix} w_k & \Delta \\ \Delta^* & w_k \end{pmatrix} \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_0 & \\ & \varepsilon_0 \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} \varepsilon_0 & \\ & \varepsilon_0 \end{pmatrix}$$

$$\begin{pmatrix} w_k & \Delta \\ \Delta^* & w_k \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} u_k \\ -v_k \end{pmatrix} \varepsilon_0$$

where to invert the matrix we used:

$$\begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix}^{-1} = \frac{1}{u_k^2 - v_k^2} \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \qquad [\text{using } u_k^2 - v_k^2 = 1]$$